

Bachelor's thesis

Optimal Boundedness Results for Degenerate Elliptic Equations

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Abstract

We study a certain integrability condition on the coefficient field of linear, nonuniformly elliptic equations which yields the local boundedness for solutions. Such an integrability condition was found by Trudinger and then improved by P. Bella and M. Schäffner. By modifying a counterexample of B. Franchi, R. Serapioni and F. Serra Cassano for the complementary condition in the spatial dimension $d = 4$, we prove the sharpness of the condition of P. Bella and M. Schäffner. Contrary to what was previously assumed by the authors, we will see that their example is suitable not only for $d \geq 4$, but also for $d = 3$ and even clarifies the borderline case of the above-mentioned condition for $d \geq 4$. The proof of sharpness is based on the proof of B. Franchi, R. Serapioni and F. Serra Cassano, with some arguments discussed in more detail.

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1 Introduction and Main Results

Consider the linear, second order, scalar elliptic equation

$$\nabla \cdot a \nabla u = 0, \quad (1.1)$$

where $d \geq 2$ and $a : \Omega \rightarrow \mathbb{R}^{d \times d}$ is a measurable matrix field on a domain $\Omega \subset \mathbb{R}^d$. Moreover, we define

$$\lambda(x) := \inf_{\xi \in \mathbb{R}^d} \frac{\xi \cdot a(x) \xi}{|\xi|^2}, \quad \mu(x) := \sup_{\xi \in \mathbb{R}^d} \frac{|a(x) \xi|^2}{\xi \cdot a(x) \xi} \quad (1.2)$$

and suppose λ and μ are nonnegative measurable functions. The equation (1.1) is often used in physics to model an equilibrium, where the solution u can be interpreted as a density of some physical quantity. Thus, it is interesting to study the properties of a solution u of (1.1) depending on the coefficient field a . Before we now list some results, note that the term *weak solution* will be defined later.

In [De 57] De Giorgi and in [Nas58] Nash proved that weak solutions of (1.1) are Hölder continuous, if λ^{-1} and μ are essentially bounded (i.e. a is uniformly elliptic). In [Tru71] Trudinger or in [MS68] Murthy and Stampacchia considered $\lambda^{-1} \in L^q(\Omega)$ and $\mu \in L^p(\Omega)$ with $p, q \in (1, \infty]$ and proved that, if $\frac{1}{p} + \frac{1}{q} < \frac{2}{d}$, where $\frac{1}{\infty} := 0$, then weak solutions of (1.1) are locally bounded. This result was improved by Bella and Schäffner [BS19] by assuming p and q satisfy the *pq-condition* $\frac{1}{p} + \frac{1}{q} < \frac{2}{d-1}$. To be more precise, they proved the following theorem.

Theorem 1.1. *Fix $d \geq 2$, a domain $\Omega \subset \mathbb{R}^d$ and $p, q \in (1, \infty]$ satisfying the pq-condition*

$$\frac{1}{p} + \frac{1}{q} < \frac{2}{d-1}. \quad (1.3)$$

Let $a : \Omega \rightarrow \mathbb{R}^{d \times d}$ be such that λ and μ given in (1.2) are nonnegative and satisfy $\lambda^{-1} \in L^q(\Omega)$, $\mu \in L^p(\Omega)$. Then, any weak solution u of (1.1) in Ω is locally bounded. More precisely, for every $\gamma > 0$ there exists $c = c(\gamma, d, p, q) \in [1, \infty)$ such that for any ball $B_R \subset \Omega$, $R > 0$, it holds that

$$\|u\|_{L^\infty(B_{R/2})} \leq c \Lambda(B_R)^{\frac{p'}{\gamma}(1+\frac{1}{\delta})} \left(\int_{B_R} |u|^\gamma \right)^{\frac{1}{\gamma}},$$

where $\delta := \min\{\frac{1}{d-1} - \frac{1}{2p}, \frac{1}{2}\} - \frac{1}{2q} > 0$, $p' := \frac{p}{p-1}$, and for every measurable set $S \subset \Omega$

$$\Lambda(S) := \left(\int_S \mu^p \right)^{\frac{1}{p}} \left(\int_S \lambda^{-q} \right)^{\frac{1}{q}}.$$

We study whether the pq -condition (1.3) is sharp. Are there coefficient fields that satisfy $\lambda^{-1} \in L^q(\Omega)$ and $\mu \in L^p(\Omega)$ with

$$\frac{1}{p} + \frac{1}{q} \geq \frac{2}{d-1}, \quad (1.4)$$

such that the weak solutions are unbounded on a ball?

The spatial dimension $d = 2$ is already discussed in [BS19], where $p = 1$ and $q = 1$ in the complementary case. To be more precise, the following proposition was proven there.

Proposition 1.2. *Fix a domain $\Omega \subset \mathbb{R}^2$. Let $a : \Omega \rightarrow \mathbb{R}^{2 \times 2}$ be measurable such that λ and μ given (1.2) are nonnegative and satisfy $\frac{1}{\lambda}, \mu \in L^1(\Omega)$. Then, there exists $c \in [1, \infty)$ such that for every weak solution u of (1.1) and for any ball $B_R \subset \Omega$*

$$\|u\|_{L^\infty(B_{R/2})} \leq c \left(R \left(\int_{B_R} \lambda^{-1} \right)^{\frac{1}{2}} \left(\int_{B_R} a \nabla u \cdot \nabla u \right)^{\frac{1}{2}} + \int_{B_R} |u| \right).$$

Thus, it remains to study $d \geq 3$. Indeed, for $d = 4$ Franchi, Serapioni and Serra Cassano constructed in [FSC98] for $\frac{1}{p} + \frac{1}{q} > \frac{2}{d-1}$ a coefficient field a such that a weak solution of (1.1) is unbounded on the unit ball. They also claim that their counterexample can be generalised to the case $d \geq 4$. In this bachelor thesis we will see how their counterexample can be modified. Moreover, it turns out that their counterexample also works for the borderline case $\frac{1}{p} + \frac{1}{q} = \frac{2}{d-1}$ and by a simple modification even for $d = 3$. However, the modification for $d = 3$ does not cover the borderline case. Thus, it is still not clear if for $\frac{1}{p} + \frac{1}{q} = \frac{2}{3-1} = 1$ weak solutions of (1.1) are locally bounded.

To be more exact, we will prove in this bachelor's thesis the following theorems.

Theorem 1.3. *Let $d = 3$, $\Omega = B_1 \subset \mathbb{R}^3$ denote the unit ball and $p, q \in (1, \infty)$ satisfy*

$$\frac{1}{p} + \frac{1}{q} > 1. \quad (1.5)$$

Then, there exists a scalar field $\omega : B_1 \rightarrow (0, \infty)$ with $\omega^{-1} \in L^q(B_1)$ and $\omega \in L^p(B_1)$ such that there exists an unbounded weak solution of

$$\nabla \cdot \omega \nabla u = 0.$$

Theorem 1.4. *Fix $d \geq 4$, let $\Omega = B_1 \subset \mathbb{R}^d$ denote the unit ball and $p, q \in (1, \infty]$ satisfy*

$$\frac{1}{p} + \frac{1}{q} \geq \frac{2}{d-1}.$$

Then, there exists a scalar field $\omega : B_1 \rightarrow (0, \infty)$ with $\omega^{-1} \in L^q(B_1)$ and $\omega \in L^p(B_1)$ such that there exists an unbounded weak solution of

$$\nabla \cdot \omega \nabla u = 0.$$

Note that in Theorem 1.3 the case $p = \infty$ or $q = \infty$ cannot occur, otherwise the inequality (1.5) cannot be satisfied. Moreover, the scalar field ω can be identified by $a = \omega I_d$, where I_d denotes the $d \times d$ identity matrix. Additionally, we obtain $\lambda = \omega$ and $\mu = \omega$, where λ and μ are as in (1.2).

This bachelor's thesis is organised as follows: In chapter 2, we clarify the meaning of weak solutions in this setting by introducing a suitable function space. We also prove the weak maximum principle and some auxiliary lemmas that we will use in chapter 3. In particular, we prove that it is sufficient to construct an unbounded subsolution for a given coefficient matrix. Finally, in chapter 3 we prove Theorem 1.3 and Theorem 1.4. We first discuss the case $d = 3$ in detail by motivating the steps of the proof. With $d \geq 4$, we proceed almost in the same way as in the case $d = 3$.

2 Auxiliary Lemmas

As in [FSC98], we will prove Theorem 1.3 and Theorem 1.4 by constructing a coefficient field and an unbounded weak subsolution instead of an unbounded weak solution. The authors claimed that this follows, because of the weak maximum principle, and do not go into this in detail. We will take a closer look at this argument.

For this we will first clarify the meaning of weak subsolution. Afterwards, we prove the weak maximum principle, such that we are able to prove that it is sufficient to construct a weak subsolution, which will be our first auxiliary lemma. The second one is just a simple lemma which we will need for some calculations later.

Definition 2.1. Let $\Omega \subset \mathbb{R}^d$ be a domain and $a : \Omega \rightarrow \mathbb{R}^{d \times d}$ be a coefficient field such that $\mu, \lambda \geq 0$ given in (1.2) satisfy $\frac{1}{\lambda}, \mu \in L^1(\Omega)$. Then, the function spaces $H^1(a, \Omega)$ and $H_0^1(a, \Omega)$ are, respectively, defined as the completion of $C^1(\Omega)$ and $C_c^1(\Omega)$ with respect to the norm $\|\cdot\|_{H^1(a, \Omega)} := \sqrt{\mathcal{L}_1(\cdot, \cdot)}$, where

$$\begin{aligned}\mathcal{L}(u, v) &:= \int_{\Omega} a(x) \nabla u(x) \cdot \nabla v(x) \, dx, \\ \mathcal{L}_1(u, v) &:= \mathcal{L}(u, v) + \int_{\Omega} \mu(x) u(x) v(x) \, dx.\end{aligned}$$

We say that u is a *weak solution* of (1.1) if and only if $u \in H^1(a, \Omega)$ and $\mathcal{L}(u, \varphi) = 0$ for every $\varphi \in H_0^1(a, \Omega)$. Moreover, we call u a *weak subsolution* if and only if $u \in H^1(a, \Omega)$ and

$$\forall \varphi \in H_0^1(a, \Omega), \varphi \geq 0 : \mathcal{L}(u, \varphi) \leq 0.$$

Note that $H^1(a, \Omega)$ and $H_0^1(a, \Omega)$ equipped with $\mathcal{L}_1$ are Hilbert spaces. Furthermore, if one wants to prove that some function $u \in H^1(a, \Omega)$ is a weak solution of (1.1), it suffices to show $\mathcal{L}(u, \varphi) = 0$ for every $\varphi \in C_c^\infty(\Omega)$, since $C_c^\infty(\Omega)$ is dense in $H_0^1(a, \Omega)$. The same holds for weak subsolutions. Finally, let us write, for simplicity, $H^1(\omega, \Omega) := H^1(\omega I_d, \Omega)$ and $H_0^1(\omega, \Omega) := H_0^1(\omega I_d, \Omega)$ if ω is a scalar field.

Weak Maximum Principle

The following two lemmas exhibit the properties of functions in $H_0^1(a, \Omega)$ required for the proof of a maximum principle. Further properties of $H^1(a, \Omega)$ and $H_0^1(a, \Omega)$ were treated by Trudinger in [Tru73] and [Tru71]. Our first lemma can also be looked up there and can be considered as a generalised chain rule.

Lemma 2.2. Let $\Omega \subset \mathbb{R}^n$ be a domain and $g : \mathbb{R} \rightarrow \mathbb{R}$ uniformly Lipschitz-continuous with $g(0) = 0$. Consider $u \in H^1(a, \Omega)$ or $u \in H_0^1(a, \Omega)$. Then, the composition $g(u)$ belongs to $H^1(a, \Omega)$ or $H_0^1(a, \Omega)$, respectively, and the chain rule applies, i.e.

$$\nabla(g(u)) = g'(u) \nabla u \quad \text{a.e. in } \Omega.$$

By Lemma 2.2 and $g(x) = x\mathbb{1}_{\{x>0\}}$ we get
Remark 2.3. Let $u \in H_0^1(a, \Omega)$. Then, $u^+ := \sup\{u, 0\} \in H_0^1(a, \Omega)$ and

$$\nabla u^+ = \nabla u \mathbb{1}_{\{u>0\}} \quad \text{a.e. in } \Omega.$$

In the following lemma we prove that the function space $H_0^1(a, \Omega)$ is a subset of a Sobolev space. For this purpose we will use Poincaré's inequality.

Theorem 2.4 (Poincaré's inequality). *Assume $\Omega \subset \mathbb{R}^d$ is open and bounded. Suppose u belongs to the Sobolev space $W_0^{1,p}(\Omega)$ for some $p \in [1, \infty]$. Then, we have the estimate*

$$\|u\|_{W^{1,p}(\Omega)} \leq C_P \|\nabla u\|_{L^p(\Omega)},$$

where the constant C_P depends only on p , d and Ω .

A proof of this theorem and more about Sobolev spaces like construction, definition and properties can be found in [Eva10] or [Sch13]. Note that we need some restrictions for the use of Poincaré's inequality, i.e. a bounded domain and the function space $H_0^1(a, \Omega)$ instead of $H^1(a, \Omega)$. But as we will see, this is sufficient for our purposes.

Lemma 2.5. *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain and λ given in (1.2) satisfy $\lambda^{-1} \in L^q(\Omega)$ for $q \in [1, \infty]$. Consider $u \in H_0^1(a, \Omega)$ and define*

$$q^* := \begin{cases} \frac{2q}{q+1} & \text{when } q \in [1, \infty), \\ 2 & \text{when } q = \infty. \end{cases}$$

Then, u belongs to the Sobolev space $W_0^{1,q^*}(\Omega)$ or, more precisely,

$$\|u\|_{W^{1,q^*}(\Omega)}^2 \leq C_P^2 \|\lambda^{-1}\|_{L^q(\Omega)} \int_{\Omega} a \nabla u \cdot \nabla u, \quad (2.1)$$

where C_P denotes the constant from Poincaré's inequality in Theorem 2.4.

Proof. It suffices to show the inequality (2.1) holds for a smooth function with compact support $u \in H^1(a, \Omega)$, because every function of $H_0^1(a, \Omega)$ or $W_0^{1,q^*}(\Omega)$ can be approximated by smooth functions with compact support with respect to their norms (see Definition 2.1 and [Sch13]). Consider $u \in C_c^\infty(\Omega) \cap H^1(a, \Omega)$. In the case $q \in [1, \infty)$ we obtain, using Hölder's inequality,

$$\|\nabla u\|_{L^{q^*}(\Omega)}^2 = \left(\int_{\Omega} \frac{\lambda^{\frac{q}{q+1}}}{\lambda^{\frac{q}{q+1}}} |\nabla u|^{\frac{2q}{q+1}} \right)^{\frac{q+1}{q}} \leq \|\lambda^{-1}\|_{L^q(\Omega)} \int_{\Omega} \lambda |\nabla u|^2 \leq \|\lambda^{-1}\|_{L^q(\Omega)} \int_{\Omega} a \nabla u \cdot \nabla u$$

and, if $q = \infty$,

$$\|\nabla u\|_{L^2(\Omega)}^2 = \int_{\Omega} \frac{\lambda}{\lambda} |\nabla u|^2 \leq \|\lambda^{-1}\|_{L^\infty(\Omega)} \int_{\Omega} \lambda |\nabla u|^2 \leq \|\lambda^{-1}\|_{L^\infty(\Omega)} \int_{\Omega} a \nabla u \cdot \nabla u.$$

By Poincaré's inequality in Theorem 2.4, we get

$$\|\lambda^{-1}\|_{L^q(\Omega)} \int_{\Omega} a \nabla u \cdot \nabla u \geq \|\nabla u\|_{L^{q^*}(\Omega)}^2 \geq C_P^{-2} \|u\|_{W^{1,q^*}(\Omega)}^2,$$

which yields estimate (2.1). □

Next, we prove the maximum principle. For this purpose, let us say a function $u \in H^1(a, \Omega)$ satisfies $u \leq m$ on $\partial\Omega$ for some $m \geq 0$ if $(u - m)^+ := \sup\{u - m, 0\}$ belongs to $H_0^1(a, \Omega)$.

Theorem 2.6 (Weak Maximum Principle). *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain, $q \in [1, \infty]$ and $a : \Omega \rightarrow \mathbb{R}^{d \times d}$ be a coefficient field such that $\mu, \lambda \geq 0$ given in (1.2) satisfy $\frac{1}{\lambda} \in L^q(\Omega)$ and $\mu \in L^1(\Omega)$. If a weak subsolution $u \in H^1(a, \Omega)$ of (1.1) satisfies $u \leq m$ on $\partial\Omega$ for some $m \geq 0$, then*

$$u \leq m \quad \text{a.e. in } \Omega.$$

Proof. We define $\varphi := (u - m)^+ = \sup\{u - m, 0\}$. The maximum principle is proven when we show $\varphi = 0$. According to Remark 2.3, φ belongs to $H_0^1(a, \Omega)$ and $\nabla\varphi = \nabla u \mathbb{1}_{\{u > 0\}}$. Since u is a weak subsolution and $\varphi \geq 0$, we obtain

$$\begin{aligned} 0 \geq \mathcal{L}(u, \varphi) &= \int_{\Omega} a \nabla u \cdot \nabla \varphi \\ &= \int_{\Omega} a \nabla u \cdot \nabla u \mathbb{1}_{\{u > 0\}} \\ &= \int_{\Omega} a \nabla \varphi \cdot \nabla \varphi \\ &\geq C_P^{-2} \|\lambda^{-1}\|_{L^q(\Omega)}^{-1} \|\varphi\|_{W^{1,q^*}(\Omega)}^2 \\ &\geq 0, \end{aligned}$$

where we used the estimate (2.1) in Lemma 2.5. Hence, we obtain $\|\varphi\|_{W^{1,q^*}(\Omega)} = 0$ which implies that $\varphi = 0$. $\square$

Auxiliary Lemmas

Now, we move on to the announced auxiliary lemmas. For this purpose, let us say the functions $u, v \in H^1(a, \Omega)$ satisfy $u = v$ on $\partial\Omega$ if $u - v$ belongs to $H_0^1(a, \Omega)$.

Lemma 2.7. *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain and λ given in (1.2) satisfy $\lambda^{-1} \in L^q(\Omega)$ for $q \in [1, \infty]$. Assume that an unbounded weak subsolution v of (1.1) and a weak solution u of (1.1) exist. If $u = v$ on $\partial\Omega$, then u is unbounded.*

Proof. Since $v - u \in H_0^1(a, \Omega)$, we have $(v - u)^+ \in H_0^1(a, \Omega)$ due to Remark 2.3. Thus, we get $v - u \leq 0$ on $\partial\Omega$. Additionally, $v - u$ is a weak subsolution of (1.1) and, according to the weak maximum principle, we obtain

$$v - u \leq 0 \quad \text{a.e. in } \Omega.$$

$\square$

Note that with this lemma it suffices to construct an unbounded weak subsolution to prove Theorem 1.3 and Theorem 1.4, but we have still to show that such a solution exists.

Finally, we prove the following simple lemma that we need for some calculations.

Lemma 2.8. *Let $\gamma > 1$. The integral*

$$\int_0^1 r^p \left(\log \frac{\gamma}{r}\right)^q dr \quad (2.2)$$

converges in case $p > -1$ for all $q \in \mathbb{R}$ and case $p = -1$ if $q < -1$.

Proof. We first rewrite the integral by a simple change of variables, i.e. $z = \log \frac{\gamma}{r}$,

$$\int_0^1 r^p \left(\log \frac{\gamma}{r}\right)^q dr = \gamma^{p+1} \int_{\log \gamma}^{\infty} e^{-z(p+1)} z^q dz. \quad (2.3)$$

Note that $\log \gamma > 0$, because $\gamma > 1$. For $p = -1$, the integral simplifies to

$$\gamma^{p+1} \int_{\log \gamma}^{\infty} z^q dz,$$

which converges for $q < -1$.

If $p > -1$, the function $f(z) := e^{-z(p+1)} z^q$ is bounded by some constant $c := c(p, q) > 0$ on $(\log \gamma, \infty)$ and it is decreasing from some point $z_0 > \log \gamma$. Therefore, the integral (2.3) can be estimated for $N \geq z_0$ by

$$\gamma^{p+1} \left(cN + \sum_{n=N}^{\infty} e^{-n(p+1)} n^q \right).$$

The series converges due to the ratio test and therefore the integral (2.2) also does. $\square$

3 Construction of the Counterexample

In this chapter, we construct counterexamples which provide a proof of Theorem 1.3 and Theorem 1.4. We proceed in a similar way as in [FSC98]. The construction consists of three parts. In the first one, we define the coefficient field which satisfies the complementary pq -condition (1.4). According to Lemma 2.7, it is sufficient to construct an unbounded subsolution and to show the existence of a certain weak solution, which happens in the second and in the third part.

3.1 Counterexample in three dimensions

Fix $d = 3$ and $p, q \in (1, \infty)$ such that

$$\frac{1}{p} + \frac{1}{q} > 1. \quad (3.1)$$

Consider $\theta \in (0, 1)$ such that

$$\frac{1}{p} > (1 - \theta) \quad \text{and} \quad \frac{1}{q} > \theta.$$

Note that we have $\theta \notin \{0, 1\}$, because of $p, q \in (1, \infty)$. Throughout the proof we write $\lesssim$ if $\leq$ holds up to a positive constant that depends only on p and q .

Construction of the Coefficient Field

As stated in Theorem 1.3, we construct the coefficient field on the unit ball B_1 . The idea of the proof is to construct a coefficient field such that a weak solution becomes unbounded around the x_1 -axis. The simplest idea is to consider coefficient fields that are independent of x_1 and depend only on the distance from $\{x_1 = 0\}$. Let us denote this distance

$$r := \sqrt{x_2^2 + x_3^2}$$

for the entire section. Obviously, we have $r \in [0, 1]$. Furthermore, it is advantageous for a coefficient field to consider a step function. Like in [FSC98], we define our coefficient field $\omega_\theta : B_1 \rightarrow \mathbb{R}^+$ by

$$\omega_\theta(x) := \begin{cases} (i+1)^\theta 4^{-2i\theta} & \text{when } r \in [\frac{1}{2}4^{-i}, 4^{-i}), \\ (i+1)^{-(1-\theta)} 4^{2i(1-\theta)} & \text{when } r \in [4^{-i-1}, \frac{1}{2}4^{-i}), \end{cases} \quad (3.2)$$

where $i \in \mathbb{N}_0$. Let us call the intervals of the form $[\frac{1}{2}4^{-i}, 4^{-i})$ *even* and those of the form $[4^{-i-1}, \frac{1}{2}4^{-i})$ *odd*. We note that the values of ω_θ alternate in the annuli between very

small and very large values. In fact, ω_θ approximates on the odd intervals a function in $L^p(B_1)$ and on the even intervals a function whose inverse is in $L^q(B_1)$, as we will see in the following two lemmas. The statements contained therein were formulated in [FSC98], but not proven.

Lemma 3.1. *Let ω_θ be as in (3.2). Then, the following inequalities hold in B_1*

$$\left(\frac{1}{\log 4} r^2 \log \frac{4}{r}\right)^\theta \leq \omega_\theta(x) \leq \left(\frac{\log 4}{r^2 \log \frac{2}{r}}\right)^{1-\theta}. \quad (3.3)$$

Proof. First, we define

$$f(r) := \left(\frac{1}{\log 4} r^2 \log \frac{4}{r}\right)^\theta \quad \text{and} \quad F(r) := \left(\frac{\log 4}{r^2 \log \frac{2}{r}}\right)^{1-\theta}. \quad (3.4)$$

A simple calculation reveals that $f' \geq 0$ and $F' \leq 0$ for $r \in (0, 1)$. In particular, F is decreasing and f is increasing on $(0, 1)$. Therefore, it is sufficient to show the inequalities (3.3) at the right end of the even and odd intervals.

Case 1. Let $r \in [\frac{1}{2}4^{-i}, 4^{-i})$. Since the function f is increasing, we have

$$f(r) \leq f(4^{-i}) = \left(\frac{1}{\log 4} 4^{-2i} \log(4^{i+1})\right)^\theta = \left(\frac{\log 4}{\log 4} (i+1) 4^{-2i}\right)^\theta = \omega_\theta(r).$$

The monotonicity of F yields the following:

$$F(r) \geq F(4^{-i}) = \left(\frac{\log 4}{4^{-2i} \log(2 \cdot 4^i)}\right)^{1-\theta} > \left(\frac{4^{2i} \log 4}{\log(4^{i+1})}\right)^{1-\theta} = \frac{4^{2i}}{i+1} \omega_\theta(r) \geq \omega_\theta(r).$$

Notice, we used the monotonicity of the logarithm and $4^{2i} \geq i+1$.

Case 2. Let $r \in [4^{-i-1}, \frac{1}{2}4^{-i})$. We obtain analogously

$$F(r) \geq F(\frac{1}{2}4^{-i}) = \left(\frac{4 \log 4}{4^{-2i} \log 4^{i+1}}\right)^{1-\theta} = \left(\frac{4^{2i}}{i+1}\right)^{1-\theta} 4^{1-\theta} \geq \omega_\theta(r),$$

where we used $4^{1-\theta} \geq 1$. Additionally, we have

$$f(r) \leq f(\frac{1}{2}4^{-i}) = \left(\frac{1}{4 \log 4} 4^{-2i} \log(2 \cdot 4^{i+1})\right)^\theta = 4^{2i(1-\theta)} 4^{-2i} \left(\frac{i}{4} + \frac{3}{8}\right)^\theta \leq \omega_\theta(r),$$

where we needed

$$4^{-2i} \left(\frac{i}{4} + \frac{3}{8}\right)^\theta \leq (i+1)^{-(1-\theta)},$$

which can easily be proven by an induction argument. $\square$

Lemma 3.2. *Let ω_θ be as in (3.2). Then, it satisfies*

$$\omega_\theta \in L^p(B_1) \quad \text{and} \quad \omega_\theta^{-1} \in L^q(B_1).$$

Proof. Let F, f be as in (3.4). Using Lemma 3.1, it suffices to show

$$F \in L^p(B_1) \quad \text{and} \quad f^{-1} \in L^q(B_1).$$

Using Fubini's theorem and radial coordinates, we obtain

$$\int_{B_1} |F(x)|^p dx \lesssim \int_0^1 dx_1 \int_0^1 r^{-2p(1-\theta)} (\log \frac{2}{r})^{-p(1-\theta)} r dr. \quad (3.5)$$

Since according to (3.1) the inequality $p < \frac{1}{1-\theta}$ holds, we have $1 - 2p(1 - \theta) > -1$ and (3.5) converges due to Lemma 2.8.

Analogously, we get

$$\int_{B_1} |f^{-1}|^q dx \lesssim \int_0^1 r^{-2q\theta} (\log \frac{4}{r})^{-q\theta} r dr.$$

and the integral converges by Lemma 2.8, since $q < \frac{1}{\theta}$ implies $-2q\theta + 1 > -1$. $\square$

Construction of the Subsolution

As discussed before, we construct an unbounded weak subsolution of

$$\nabla \cdot (\omega_\theta \nabla u) = 0. \quad (3.6)$$

We define $v : B_1 \rightarrow \mathbb{R}^+$

$$v(x) := e^{\alpha x_1} \phi(r), \quad (3.7)$$

where $\phi : (0, 1) \rightarrow \mathbb{R}^+$

$$\phi(r) := \begin{cases} i + \frac{\eta_i}{\log 2} \log \frac{4^{-i}}{r} & \text{when } r \in [\frac{1}{2}4^{-i}, 4^{-i}), \\ i + 1 + (\eta_i - 1)(4^{i+1}r - 1)^2 & \text{when } r \in [4^{-i-1}, \frac{1}{2}4^{-i}), \end{cases} \quad (3.8)$$

$$\alpha := \sqrt{\frac{12}{\log 2}}, \quad \eta_i := \frac{2}{2 + (2 \log 2)^{-1}(i + 1)4^{-2i}}. \quad (3.9)$$

These definitions are similar to those in [FSC98]. Only α , η_i and ϕ on the even intervals are adapted to the spatial dimension $d = 3$.

Like in [FSC98], we divide the proof that v is a weak subsolution of (3.6) into three lemmas. In the first one, we show $v \in H^1(\omega, B_1)$. Then, we prove that v is a weak subsolution in a positive distance away from the x_1 -axis. Arguing that the contributions from a tiny neighborhood of the axis $\{x_1 = 0\}$ to the integral in the weak formulation is negligible, we conclude that v is a subsolution in B_1 .

Before we begin with the proof, let us take a closer look at v . The number α and η_i in (3.9) are especially important for the second step. Moreover, it decreases on the interval $[4^{-i-1}, 4^{-i}]$ from $i + 1$ to i for $i \in \mathbb{N}_0$. Hence, ϕ is unbounded, because it becomes infinite as $r \rightarrow 0^+$. Furthermore, we notice that ϕ equals $i + \eta_i$ at $r = \frac{1}{2}4^{-i}$. Thus, ϕ , and therefore v , is continuous in $B_1 \setminus \{x_1 = 0\}$.

Finally, unlike in [FSC98], we will need estimates for ω_θ and for the derivatives

$$\begin{aligned}\phi'(r) &= \begin{cases} -\frac{\eta_i}{r \log 2} & \text{when } r \in (\frac{1}{2}4^{-i}, 4^{-i}), \\ 2(\eta_i - 1)4^{i+1}(4^{i+1}r - 1) & \text{when } r \in (4^{-i-1}, \frac{1}{2}4^{-i}), \end{cases} \\ \phi''(r) &= \begin{cases} \frac{\eta_i}{r^2 \log 2} & \text{when } r \in (\frac{1}{2}4^{-i}, 4^{-i}), \\ 2(\eta_i - 1)4^{2i+2} & \text{when } r \in (4^{-i-1}, \frac{1}{2}4^{-i}). \end{cases}\end{aligned}$$

Lemma 3.3. *Let ω_θ be as in (3.2) and ϕ be as in (3.8). Then,*

$$\begin{aligned}\omega_\theta(r) &\lesssim \begin{cases} r^{2\theta}(\log \frac{4}{r})^\theta & \text{when } r \in [\frac{1}{2}4^{-i}, 4^{-i}), \\ r^{-2(1-\theta)}(\log \frac{2}{r})^{-(1-\theta)} & \text{when } r \in [4^{-i-1}, \frac{1}{2}4^{-i}), \end{cases} \\ |\phi(r)| &\lesssim \begin{cases} \log \frac{4}{r} & \text{when } r \in (\frac{1}{2}4^{-i}, 4^{-i}), \\ \log \frac{2}{r} & \text{when } r \in (4^{-i-1}, \frac{1}{2}4^{-i}), \end{cases} \\ |\phi'(r)| &\lesssim \begin{cases} r^{-1} & \text{when } r \in (\frac{1}{2}4^{-i}, 4^{-i}), \\ r \log \frac{2}{r} & \text{when } r \in (4^{-i-1}, \frac{1}{2}4^{-i}). \end{cases}\end{aligned}$$

Proof. Fix $i \in \mathbb{N}_0$ and $r \in (\frac{1}{2}4^{-i}, 4^{-i})$. Since $i + 1 \leq \log \frac{4}{r}$ and $4^{-i} \leq 2r$, we get

$$\omega_\theta(r) = (i + 1)^\theta 4^{-2i\theta} \lesssim r^{2\theta} (\log \frac{4}{r})^\theta.$$

Additionally, we obtain

$$|\phi(r)| = i + \frac{\eta_i}{\log 2} \log \frac{4^{-i}}{r} \leq i + 1 \leq \log \frac{4}{r}$$

and

$$|\phi'(r)| = \frac{\eta_i}{r \log 2} \leq \frac{1}{r \log 2} \lesssim \frac{1}{r},$$

where we used $\eta_i \in (0, 1)$ and $\frac{1}{\log 2} \log \frac{4^{-i}}{r} \in [0, 1]$.

Now, fix $r \in (4^{-i-1}, \frac{1}{2}4^{-i})$. Since $i + 1 \leq \log \frac{2}{r}$ and $4^{-i} \leq 4r$, we get

$$\omega_\theta(r) = (i + 1)^{-(1-\theta)} 4^{2i(1-\theta)} \lesssim r^{-2(1-\theta)} (\log \frac{2}{r})^{-(1-\theta)}.$$

Using $(\eta_i - 1) \in (-1, 0)$ and $(4^{i+1}r - 1)^2 \geq 0$, we obtain

$$|\phi(r)| = i + 1 + (\eta_i - 1)(4^{i+1}r - 1)^2 \leq i + 1 \leq \log \frac{2}{r}.$$

Now, we must argue for $\phi'(r)$ a bit differently than before. Using the definition of η_i in (3.9), we get

$$|\phi'(r)| = 2 \frac{(2 \log 2)^{-1}(i + 1)4^{-2i}}{2 + (2 \log 2)^{-1}(i + 1)4^{-2i}} 4^{i+1}(4^{i+1}r - 1) \lesssim (i + 1)4^{-2i}4^{i+1} \lesssim r \log \frac{2}{r},$$

where we needed $(4^{i+1}r - 1) \in [0, 1]$. □

The following lemmas were also shown in [FSC98], but we will prove them in more detail.

Lemma 3.4. *Let v be as in (3.7). Then, $v \in H^1(\omega_\theta, B_1)$.*

Proof. To show $v \in H^1(\omega_\theta, B_1)$, it is sufficient to approximate it with Lipschitz functions with respect to the $H^1(\omega_\theta, B_1)$ -norm, since every Lipschitz function can be approximated by $C^\infty(B_1)$ functions. Define for $n \in \mathbb{N}$

$$\phi_n(r) := \phi(r)\mathbb{1}_{\{r \geq 4^{-n}\}} + n\mathbb{1}_{\{r < 4^{-n}\}} \quad \text{and} \quad v_n(x) := e^{\alpha x_1} \phi_n(r).$$

At this point, it was claimed in [FSC98] that $v_n \rightarrow v$ in $H^1(\omega_\theta, B_1)$, which we will now prove.

Each function v_n is Lipschitz-continuous and v_n and ∇v_n converge pointwise to v and ∇v , respectively. Moreover,

$$\begin{aligned} \|v - v_n\|_{H^1(\omega_\theta, B_1)} &= \int_{B_1} \omega_\theta(r) (|\nabla v(x) - \nabla v_n(x)|^2 + |v(x) - v_n(x)|^2) dx \\ &= \int_{B_1} e^{2\alpha x_1} \omega_\theta(r) \left((\alpha^2 + 1)|\phi(r) - n|^2 + |\phi'(r)|^2 \right) \mathbb{1}_{\{r < 4^{-n}\}}(r) dx \\ &\lesssim \int_0^1 \omega_\theta(r) (|\phi(r)|^2 + |\phi'(r)|^2) r dr, \end{aligned} \tag{3.10}$$

where we needed

$$\begin{aligned} |\nabla v(x) - \nabla v_n(x)| &= |\partial_1 v(x) - \partial_1 v_n(x)| + |\partial_r v(x) - \partial_r v_n(x)| \\ &= \alpha e^{2\alpha x_1} |\phi(r) - \phi_n(r)| + e^{2\alpha x_1} |\phi'(r) - \phi'_n(r)|, \end{aligned}$$

Fubini's theorem, radial coordinates and $\phi(r) \geq n$ if $r < 4^{-n}$.

If (3.10) converges, the claim follows by the dominated convergence theorem. We write

$$\begin{aligned} &\int_0^1 \omega_\theta(r) (|\phi(r)|^2 + |\phi'(r)|^2) r dr \\ &= \sum_{i=0}^{\infty} \int_{\frac{1}{2}4^{-i}}^{4^{-i}} \omega_\theta(r) (|\phi(r)|^2 + |\phi'(r)|^2) r dr + \int_{4^{-i-1}}^{\frac{1}{2}4^{-i}} \omega_\theta(r) (|\phi(r)|^2 + |\phi'(r)|^2) r dr \end{aligned}$$

and use Lemma 3.3 to get

$$\begin{aligned} \sum_{i=0}^{\infty} \int_{\frac{1}{2}4^{-i}}^{4^{-i}} \omega_\theta(r) |\phi(r)|^2 r dr &\lesssim \int_0^1 (\log \frac{4}{r})^{2+\theta} r^{1+2\theta} dr, \\ \sum_{i=0}^{\infty} \int_{\frac{1}{2}4^{-i}}^{4^{-i}} \omega_\theta(r) |\partial_r \phi(r)|^2 r dr &\lesssim \int_0^1 (\log \frac{4}{r})^\theta r^{-1+2\theta} dr, \\ \sum_{i=0}^{\infty} \int_{4^{-i-1}}^{\frac{1}{2}4^{-i}} \omega_\theta(r) |\phi(r)|^2 r dr &\lesssim \int_0^1 (\log \frac{2}{r})^{1+\theta} r^{-1+2\theta} dr, \\ \sum_{i=0}^{\infty} \int_{4^{-i-1}}^{\frac{1}{2}4^{-i}} \omega_\theta(r) |\partial_r \phi(r)|^2 r dr &\lesssim \int_0^1 (\log \frac{2}{r})^{1+\theta} r^{1+2\theta} dr. \end{aligned}$$

Since $\theta > 0$, all these integrals converge due to Lemma 2.8. $\square$

Lemma 3.5. *Let v be as in (3.7) and $\gamma_{2i} := 4^{-i}$, $\gamma_{2i+1} := \frac{1}{2}4^{-i}$ for $i \in \mathbb{N}_0$. Then, v is a weak subsolution of (3.6) in each cylindrical shell $B_1 \cap \{\gamma_N < r < 1\}$ with $N \in \mathbb{N}$.*

Proof. Fix $N \in \mathbb{N}$ and define $\Omega := B_1 \cap \{\gamma_N < r < 1\}$. Furthermore, let $\varphi \in C_c^\infty(\Omega)$ be nonnegative. Additionally, we denote the cylindrical shells with

$$M_{2i} := B_1 \cap \left\{ \frac{1}{2}4^{-i} < r < 4^{-i} \right\} \quad \text{and} \quad M_{2i+1} := B_1 \cap \left\{ 4^{-i-1} < r < \frac{1}{2}4^{-i} \right\},$$

the left-sided weighted derivative with $(\omega_\theta \phi')(r^-)$ and the right-sided weighted derivative with $(\omega_\theta \phi')(r^+)$. Since ω_θ is constant on each M_k , we obtain via integration by parts

$$\begin{aligned} \int_{B_1} \omega_\theta \nabla v \cdot \nabla \varphi &= \sum_{k=0}^N \int_{M_k} \omega_\theta \nabla v \cdot \nabla \varphi \\ &= \sum_{k=0}^N - \int_{M_k} \omega_\theta \Delta v \varphi + \int_{\partial M_k} \omega_\theta \nabla v \cdot \vec{n} \varphi \\ &= \sum_{k=0}^N - \int_{M_k} \omega_\theta \Delta v \varphi \\ &\quad + \sum_{k=1}^{N-1} \int_{B_1 \cap \{r=\gamma_k\}} e^{\alpha x_1} ((\omega_\theta \phi')(\gamma_k^-) - (\omega_\theta \phi')(\gamma_k^+)) \varphi(x) \, dx, \end{aligned}$$

where $\vec{n}$ denotes the unit surface normal. Since φ has compact support in Ω , the boundary integrals on the sets $B_1 \cap \{r=1\}$ and $B_1 \cap \{r=\gamma_N\}$ vanish.

Thus, we have proven what was claimed in [FSC98], i.e. it is sufficient to prove

- (i) $-\Delta v \leq 0$ in M_k ,
- (ii) $(\omega_\theta \phi')(\gamma_k^-) - (\omega_\theta \phi')(\gamma_k^+) \leq 0$ for every k .

Proof of (i). Using spherical transformation, we obtain

$$-\Delta v(x) = -e^{\alpha x_1} \left(\alpha^2 \phi(r) + \frac{1}{r} \phi'(r) + \phi''(r) \right),$$

hence $-\Delta v \leq 0$ in M_k provides

$$\alpha^2 \phi(r) + \frac{1}{r} \phi'(r) + \phi''(r) \geq 0.$$

For $r \in (\frac{1}{2}4^{-i}, 4^{-i})$, we get

$$\alpha^2 \phi(r) + \frac{1}{r} \phi'(r) + \phi''(r) = \alpha^2 \left(i + \frac{\eta_i}{\log 2} \log \frac{4^{-i}}{r} \right) + \frac{1}{r} \left(-\frac{\eta_i}{r \log 2} \right) + \frac{\eta_i}{r^2 \log 2} \geq \alpha^2 i \geq 0,$$

where the second and the third term cancel each other, and we used $\frac{\eta_i}{\log 2} \log \frac{4^{-i}}{r} \geq 0$. This should be no surprise, since the second and the third term are exactly planar laplacian,

3 Construction of the Counterexample

whereas $\Phi(x) = C_0 + C_1 \log r$, i.e. modulo constants Φ is the fundamental solution of $-\Delta u = 0$ in the plane (see [Eva10]).

Now, fix $r \in (4^{-i-1}, \frac{1}{2}4^{-i})$. Then, it follows that

$$\begin{aligned} & \alpha^2 \phi(r) + \frac{1}{r} \phi'(r) + \phi''(r) \\ &= 2(\eta_i - 1)4^{2i+2} \left(\frac{\alpha^2(i+1)}{2(\eta_i - 1)4^{2i+2}} + \frac{\alpha^2(4^{i+1}r - 1)^2}{2 \cdot 4^{2i+2}} + \frac{1}{r}(r - 4^{-i-1}) + 1 \right). \end{aligned} \quad (3.11)$$

Furthermore, we get

$$\begin{aligned} & \frac{\alpha^2(i+1)}{2(\eta_i - 1)4^{2i+2}} + \frac{\alpha^2(4^{i+1}r - 1)^2}{2 \cdot 4^{2i+2}} + \frac{1}{r}(r - 4^{-i-1}) + 1 \\ &= 2 - \frac{4^{-i-1}}{r} + \frac{\alpha^2}{2 \cdot 4^{2i+2}} \left(\frac{i+1}{\eta_i - 1} + (4^{i+1}r - 1)^2 \right) \\ &\leq \frac{3}{2} - \frac{\alpha^2}{2 \cdot 4^{2i+2}} \left(\frac{i+1}{1 - \eta_i} - 1 \right) \\ &= \frac{3}{2} - \frac{\alpha^2}{2 \cdot 4^{2i+2}} \left(\frac{(i+1)(2 + (2 \log 2)^{-1}(i+1)4^{-2i})}{(2 \log 2)^{-1}(i+1)4^{-2i}} - \frac{(2 \log 2)^{-1}4^{-2i}}{(2 \log 2)^{-1}4^{-2i}} \right) \\ &= \frac{3}{2} - \frac{\alpha^2}{2 \cdot 4^{2i+2}} \frac{2 + (2 \log 2)^{-1}i4^{-2i}}{(2 \log 2)^{-1}4^{-2i}} \\ &= \frac{3}{2} - \frac{\alpha^2 \log 2}{8} - \frac{\alpha^2 i}{2 \cdot 4^{2i+2}} \\ &= -\frac{\alpha^2}{2} i 4^{-2i-2} \leq 0, \end{aligned}$$

where we used $-\frac{1}{r} \leq -2 \cdot 4^i$, $(4^{i+1}r - 1)^2 \leq 1$ and the definitions of η_i and α as in (3.9). This and $(\eta_i - 1) \leq 0$ imply that the term (3.11) is greater than 0. In particular, note that α was defined so that the first and second summand in the penultimate line cancel each other.

Proof of (ii). Consider $r = 4^{-i}$. Then, the right-sided derivative satisfies

$$\phi'((4^{-i})^+) = \phi'((4^{-(i-1)-1})^+) = 2(\eta_{i-1} - 1)4^i(4^i 4^{-i} - 1) = 0$$

and the left-sided derivative is

$$\phi'((4^{-i})^-) = -\frac{\eta_i}{4^{-i} \log 2} < 0.$$

Thus, $(\omega_\theta \phi')((4^{-i})^-) - (\omega_\theta \phi')((4^{-i})^+) \leq 0$.

Now, fix $r = \frac{1}{2}4^{-i}$. Then, the right-sided weighted derivative satisfies

$$(\omega_\theta \phi')\left(\left(\frac{1}{2}4^{-i}\right)^+\right) = -\frac{2\eta_i}{\log 2} 4^i (i+1)^\theta 4^{-2i\theta}$$

and the left-sided weighted derivative is

$$\begin{aligned}
 (\omega_\theta \phi') \left(\left(\frac{1}{2} 4^{-i} \right)^- \right) &= 2(\eta_i - 1) 4^{i+1} \left(\frac{1}{2} 4 - 1 \right) (i+1)^{-(1-\theta)} 4^{2i(1-\theta)} \\
 &= -2 \frac{(2 \log 2)^{-1} (i+1) 4^{-2i}}{2 + (2 \log 2)^{-1} (i+1) 4^{-2i}} 4^{i+1} (i+1)^{-(1-\theta)} 4^{2i(1-\theta)} \\
 &= -\frac{2\eta_i}{\log 2} 4^i (i+1)^\theta 4^{-2i\theta} \\
 &= (\omega_\theta \phi') \left(\left(\frac{1}{2} 4^{-i} \right)^+ \right).
 \end{aligned}$$

At this point we see that η_i is defined so that $(\omega_\theta \phi') \left(\left(\frac{1}{2} 4^{-i} \right)^- \right) = (\omega_\theta \phi') \left(\left(\frac{1}{2} 4^{-i} \right)^+ \right)$. $\square$

Lemma 3.6. *Let v be as in (3.7). Then, v is a weak subsolution of (3.6) in B_1 .*

Proof. We define the cut off function $\psi_n : \mathbb{R}^+ \rightarrow [0, 1]$ such that $\psi_n \in C^1([0, 1])$, $\psi_n(r) = 0$ for $0 \leq r \leq \frac{1}{2} 4^{-n}$, $\psi_n(r) = 1$ for $4^{-n} \leq r$ and $\psi'_n(r) \leq 4^{n+1}$. Consider a nonnegative $\varphi \in C_c^\infty(B_1)$. Then, we get

$$\int_{B_1} \nabla v \cdot \nabla \varphi = \int_{B_1} \omega_\theta \nabla v \cdot \nabla ((1 - \psi_n) \varphi) + \int_{B_1} \omega_\theta \nabla v \cdot \nabla (\psi_n \varphi).$$

Since $\psi_n \phi$ has compact support with its support being a positive distance away from $\{x_1 = 0\}$, the second integral is negative according to Lemma 3.5. In the following, we argue that $\int_{B_1} \omega_\theta \nabla v \cdot \nabla ((1 - \psi_n) \varphi)$ is negligible, i.e. it vanishes as $n \rightarrow \infty$. This then completes the proof.

We get

$$\begin{aligned}
 \left| \int_{B_1} \omega_\theta \nabla v \cdot \nabla ((1 - \psi_n) \varphi) \right| &\leq \int_{B_1 \cap \{r < 4^{-n}\}} \omega_\theta |1 - \psi_n| |\nabla v \cdot \nabla \varphi| \\
 &\quad + \int_{M_{2n}} \omega_\theta |\varphi| |\nabla v \cdot \nabla \psi_n|,
 \end{aligned} \tag{3.12}$$

where $M_{2n} = B_1 \cap \{\frac{1}{2} 4^{-n} < r < 4^{-n}\}$. According to the dominated convergence theorem, the first integral in (3.12) converges to 0 as $n \rightarrow \infty$, since $\omega_\theta |1 - \psi_n| |\nabla v \cdot \nabla \varphi|$ converges pointwise to 0, and is dominated by $\omega_\theta |\nabla v \cdot \nabla \varphi|$, where, using Hölder inequality, we get

$$\int_{B_1} \omega_\theta |\nabla v \cdot \nabla \varphi| \leq \|\nabla \varphi\|_\infty \int_{B_1} \sqrt{\omega_\theta} \sqrt{\omega_\theta} |\nabla v| \leq \|\nabla \varphi\|_\infty \|\omega_\theta\|_1^{1/2} \|v\|_{H^1(\omega_\theta, B_1)} < \infty,$$

where we define $\|\cdot\|_p := \|\cdot\|_{L^p(B_1)}$ for $p \in [1, \infty]$.

We denote the radial unit vector with $\vec{e}_r$. Then, we get for the second integral in (3.12)

$$\begin{aligned}
 \int_{M_{2n}} |\varphi| |\nabla v \cdot \nabla \psi_n| \omega_\theta &\leq \|\varphi\|_\infty \int_{M_{2n}} \omega_\theta |\nabla v \cdot 4^{n+1} \vec{e}_r| \\
 &\lesssim \|\varphi\|_\infty \int_{\frac{1}{2}4^{-n}}^{4^{-n}} 4^{n+1} \omega_\theta(r) |\phi'| r \, dr \\
 &\lesssim \|\varphi\|_\infty \int_{\frac{1}{2}4^{-n}}^{4^{-n}} (n+1)^\theta 4^{n(1-2\theta)} \, dr \\
 &= \frac{1}{2} \|\varphi\|_\infty (n+1)^\theta 4^{-2n\theta},
 \end{aligned}$$

where we used Fubini's theorem, radial coordinates, the definition of ω_θ in (3.2) and the estimate $|\phi'(r)| \lesssim \frac{1}{r}$ from Lemma 3.3. Since $\theta > 0$, the sequence $(n+1)^\theta 4^{-2n\theta}$ converges to 0 as $n \rightarrow \infty$. Therefore, the second integral in (3.12) also vanishes. $\square$

Existence of a solution

In the following, we solve the following Dirichlet problem

$$\begin{aligned}
 -\nabla \cdot \omega_\theta \nabla u &= 0 & \text{in } B_1, \\
 u &= v & \text{on } \partial B_1,
 \end{aligned} \tag{3.13}$$

where v is the weak subsolution constructed above. This concludes the proof of Theorem 1.3 using Lemma 2.7. As mentioned in chapter 2, the existence of a weak solution of (3.13) was not proven in [FSC98]. For that, we argue as in the construction for the subsolution. First, we show that on a positive distance away from the line $\{x_1 = 0\}$ a weak solution of (3.13) exists, i.e. a weak solution $\tilde{u}_n$ exists for

$$\begin{aligned}
 -\nabla \cdot \omega_\theta \nabla \tilde{u}_n &= 0 & \text{in } \Omega_n, \\
 \tilde{u}_n &= v & \text{on } \partial\Omega_n,
 \end{aligned} \tag{3.14}$$

where $\Omega_n := B_1 \cap \{\frac{3}{8}4^{-n} < r < 1\}$ and $n \in \mathbb{N}$. Afterwards, we prove that the sequence $(\tilde{u}_n)_n$ converges to the desired weak solution of (3.13).

Lemma 3.7. *Let v be as in (3.7). Then, a weak solution $\tilde{u}_n \in H^1(\omega_\theta, \Omega_n)$ of (3.14) exists for each $n \in \mathbb{N}$. Additionally, the following estimate holds:*

$$\|\tilde{u}_n - v\|_{H^1(\omega_\theta, \Omega_n)} \leq \|v\|_{H^1(\omega_\theta, \Omega)}. \tag{3.15}$$

Proof. The main idea is that a positive distance away from the line $\{x_1 = 0\}$ the coefficient field ω_θ is bounded away from 0 and from above. Then, the common theory yields the existence of a weak solution of (3.14). However, for the estimate (3.15) we must argue a bit differently.

Note that the $H^1(\omega_\theta, \Omega_n)$ -norm is equivalent to the $H^1(\Omega_n)$ -norm, because of

$$\|\omega_\theta^{-1}\|_{L^\infty(\Omega_n)}^{-\frac{1}{2}} \|u\|_{H^1(\Omega_n)} \leq \|u\|_{H^1(\omega_\theta, \Omega_n)} \leq \|\omega_\theta\|_{L^\infty(\Omega_n)}^{\frac{1}{2}} \|u\|_{H^1(\Omega_n)}, \tag{3.16}$$

and that $H_0^1(\omega_\theta, \Omega_n) \subset H_0^1(\Omega_n)$ due to Lemma 2.5. Let us define $c_n := \|\omega_\theta^{-1}\|_{L^\infty(\Omega_n)}^{-1}$ and $C_n := \|\omega_\theta\|_{L^\infty(\Omega_n)}$.

Now, fix $n \in \mathbb{N}$, set $H := H_0^1(\omega_\theta, \Omega_n)$ and consider the bilinear form

$$\begin{aligned} \mathcal{L} : H \times H &\rightarrow \mathbb{R} \\ (u, v) &\mapsto \int_{\Omega_n} \omega_\theta \nabla u \cdot \nabla v. \end{aligned}$$

The mapping $\mathcal{L}$ is continuous and, using the equivalence to the $H^1(\Omega_n)$ -norm (3.16) and Poincaré's inequality in Theorem 2.4, we obtain

$$\mathcal{L}(u, u) \geq c_n \|\nabla u\|_{L^2(\Omega_n)}^2 \geq c_n C_P^{-2} \|u\|_{H^1(\Omega_n)}^2 \geq C_n c_n C_P^{-2} \|u\|_H^2,$$

where C_P denotes the constant in Poincaré's inequality. Note that $H_0^1(\Omega) := W_0^{1,2}(\Omega)$ for some domain $\Omega \subset \mathbb{R}^n$ (see [Sch13] or [Eva10]).

According to the Lax-Milgram theorem (see [Sch13], Theorem 6.8), a unique $v_n \in H$ exists such that

$$\mathcal{L}(v_n, \varphi) = -\mathcal{L}(v, \varphi), \quad \forall \varphi \in H$$

and

$$\|v_n\|_H \leq \|\mathcal{L}(v, \cdot)\|_{H'},$$

since $-\mathcal{L}(v, \cdot) \in H'$. Then, $\tilde{u}_n = v_n + v$ is a weak solution of (3.14) and it satisfies $\tilde{u}_n - v \in H_0^1(\omega_\theta, \Omega)$. Moreover, we get

$$\|\tilde{u}_n - v\|_{H^1(\omega_\theta, \Omega_n)} \leq \sup_{\substack{\varphi \in H, \\ \|\varphi\|_H \leq 1}} \int_{\Omega_n} \omega_\theta \nabla v \cdot \nabla \varphi \leq \sup_{\substack{\varphi \in H, \\ \|\varphi\|_H \leq 1}} \|v\|_{H^1(\omega_\theta, B_1)} \|\varphi\|_H \leq \|v\|_{H^1(\omega_\theta, B_1)}.$$

□

Lemma 3.8. *The Dirichlet problem (3.13) has a weak solution u .*

Proof. Using Lemma 3.7, we obtain a sequence $(\tilde{u}_n)_n$ of weak solutions of the Dirichlet problem (3.14) for each $n \in \mathbb{N}$. We extend these functions as follows:

$$u_n := v + (\tilde{u}_n - v) \mathbb{1}_{\Omega_n}.$$

Since $(\tilde{u}_n - v) \mathbb{1}_{\Omega_n}$ is in $H_0^1(\omega_\theta, B_1)$ and $v \in H^1(\omega_\theta, B_1)$ due to Lemma 3.4, u_n belongs to $H^1(\omega_\theta, B_1)$. Using estimate (3.15), we get for every $n \in \mathbb{N}$

$$\|u_n\|_{H^1(\omega_\theta, B_1)} \leq 2\|v\|_{H^1(\omega_\theta, B_1)}. \quad (3.17)$$

Thus, a function $u \in H^1(\omega_\theta, B_1)$ exists such that

$$u_n \rightharpoonup u \quad \text{in } H^1(\omega_\theta, B_1).$$

Now, we have a candidate for a weak solution. Consider a $\varphi \in C_c^\infty(B_1)$. Then, the weak convergence implies

$$\int_{B_1} \omega_\theta \nabla u_n \cdot \nabla \varphi \xrightarrow{n \rightarrow \infty} \int_{B_1} \omega_\theta \nabla u \cdot \nabla \varphi.$$

The claim is proven if we show that $\int_{B_1} \omega_\theta \nabla u_n \cdot \nabla \varphi$ also converges to 0.

Like in the proof of Lemma 3.6, we define the cut off function $\psi_n : \mathbb{R}^+ \rightarrow [0, 1]$ such that $\psi_n \in C^1(\mathbb{R})$, $\psi_n(r) = 0$ for $0 \leq r \leq \frac{1}{2}4^{-n}$, $\psi_n(r) = 1$ for $4^{-n} \leq r$ and $\psi'_n(r) \leq 4^{n+1}$. Then, we get

$$\begin{aligned} \int_{B_1} \omega_\theta \nabla u_n \cdot \nabla \varphi &= \\ &= \int_{B_1 \cap \{r < \frac{3}{8}4^{-n}\}} \omega_\theta \nabla v \cdot \nabla \varphi + \int_{\Omega_n} \omega_\theta \nabla \tilde{u}_n \cdot \nabla ((1 - \psi_n)\varphi) + \int_{\Omega_n} \omega_\theta \nabla \tilde{u}_n \cdot \nabla (\psi_n \varphi). \end{aligned}$$

The first integral converges to 0 as shown in the proof of Lemma 3.6 and the last one vanishes, because $\tilde{u}_n$ is a weak solution in Ω_n and $\psi_n \varphi$ has compact support in Ω_n .

Let us observe the second integral which can be decomposed as follows:

$$\begin{aligned} \left| \int_{\Omega_n} \omega_\theta \nabla \tilde{u}_n \cdot \nabla ((1 - \psi_n)\varphi) \right| &\leq \int_{B_1 \cap \{r < 4^{-n}\}} \omega_\theta |1 - \psi_n| |\nabla \tilde{u}_n \cdot \nabla \varphi| \\ &\quad + \int_{M_{2n}} \omega_\theta |\varphi| |\nabla \tilde{u}_n \cdot \nabla \psi_n|, \end{aligned} \quad (3.18)$$

where $M_{2n} = B_1 \cap \{\frac{1}{2}4^{-n} < r < 4^{-n}\}$. According to the dominated convergence theorem, the first integral in (3.18) converges to 0 as $n \rightarrow \infty$, since $\omega_\theta |1 - \psi_n| |\nabla \tilde{u}_n \cdot \nabla \varphi|$ converges pointwise to 0, and is dominated by $\omega_\theta |\nabla \tilde{u}_n \cdot \nabla \varphi|$, where, using Hölder inequality and estimate (3.17), we obtain

$$\int_{B_1} \omega_\theta |\nabla \tilde{u}_n \cdot \nabla \varphi| \leq \|\nabla \varphi\|_\infty \int_{B_1} \sqrt{\omega_\theta} \sqrt{\omega_\theta} |\nabla u_n| \leq 2 \|\nabla \varphi\|_\infty \|\omega_\theta\|_{L^1}^{1/2} \|v\|_{H^1(\omega_\theta, B_1)}.$$

For the second integral in (3.18) we get

$$\begin{aligned} \int_{M_{2n}} \omega_\theta |\varphi| |\nabla \tilde{u}_n \cdot \nabla \psi_n| &\leq \|\varphi\|_\infty \|u_n\|_{H^1(\omega_\theta, B_1)} \left(\int_{M_{2n}} \omega_\theta |\nabla \psi_n|^2 \right)^{\frac{1}{2}} \\ &\lesssim 2 \|\varphi\|_\infty \|v\|_{H^1(\omega_\theta, B_1)} \left(\int_{\frac{1}{2}4^{-n}}^{4^{-n}} \omega_\theta(r) 4^{2n+2} r \, dr \right)^{\frac{1}{2}} \\ &\lesssim 2 \|\varphi\|_\infty \|v\|_{H^1(\omega_\theta, B_1)} \left(\int_{\frac{1}{2}4^{-n}}^{4^{-n}} (n+1)^\theta 4^{n(1-2\theta)} \, dr \right)^{\frac{1}{2}} \\ &= \|\varphi\|_\infty \|v\|_{H^1(\omega_\theta, B_1)} (n+1)^{\frac{\theta}{2}} 4^{-n\theta}, \end{aligned}$$

where we used Hölder's inequality, estimate (3.17), Fubini's theorem, radial coordinates, the definition of ω_θ in (3.2) and $r \leq 2 \cdot 4^{-n}$, if $r \in (\frac{1}{2}4^{-n}, 4^{-n})$. Since $\theta > 0$, the sequence $(n+1)^{\frac{\theta}{2}} 4^{-n\theta}$ converges to 0 as $n \rightarrow \infty$. Therefore, the second integral in (3.18) also vanishes. $\square$

3.2 Counterexample for four and more dimensions

The proof of Theorem 1.4 is similar to the proof of Theorem 1.3 given in the previous section and only small differences apply. We will repeat the proof of the most lemmas for better readability and point out the main differences in the beginning of each proof.

Fix $d \geq 4$ and $p, q \in (1, \infty]$ such that

$$\frac{1}{p} + \frac{1}{q} \geq \frac{2}{d-1}.$$

Consider $\theta \in [0, 1]$ such that

$$\frac{1}{p} \geq \frac{2(1-\theta)}{d-1} \quad \text{and} \quad \frac{1}{q} \geq \frac{2\theta}{d-1}.$$

Throughout the proof we write $\lesssim$ if $\leq$ holds up to a positive constant that depends only on d, p and q . As already mentioned, we also allow the borderline case in comparison to the case $d = 3$.

Construction of the Coefficient Field

As stated in Theorem 1.3, we construct the coefficient field on the unit ball B_1 . Let us denote the distance to the x_1 -axis

$$r := \sqrt{x_2^2 + \cdots + x_d^2}$$

for the entire section. Obviously, we have $r \in [0, 1]$. As we have done before, we define our coefficient field $\omega_\theta : B_1 \rightarrow \mathbb{R}^+$ with

$$\omega_\theta(x) := \begin{cases} (i+1)^\theta 4^{-2i\theta} & \text{when } r \in [\frac{1}{2}4^{-i}, 4^{-i}), \\ (i+1)^{-(1-\theta)} 4^{2i(1-\theta)} & \text{when } r \in [4^{-i-1}, \frac{1}{2}4^{-i}), \end{cases} \quad (3.19)$$

where $i \in \mathbb{N}_0$. As the definition is identical to the previous one, the same estimates hold.

Lemma 3.9. *Let ω_θ be as in (3.19). Then, the following inequalities hold in B_1*

$$\left(\frac{1}{\log 4} r^2 \log \frac{4}{r} \right)^\theta \leq \omega_\theta(x) \leq \left(\frac{\log 4}{r^2 \log \frac{2}{r}} \right)^{1-\theta}.$$

Proof. See proof of Lemma 3.1. □

Combining Lemma 3.9 with Lemma 2.8 we get

Lemma 3.10. *Let ω_θ be as in (3.19). Then, it satisfies*

$$\omega_\theta \in L^p(B_1), \quad \omega_\theta^{-1} \in L^q(B_1).$$

Proof. In this proof we must additionally discuss several cases which do not occur in Lemma 3.2, i.e. $p, q = \infty$, $p = \frac{d-1}{2(1-\theta)}$ and $q = \frac{d-1}{2\theta}$. We define

$$F(r) := \left(\frac{\log 4}{r^2 \log \frac{2}{r}} \right)^{1-\theta} \quad \text{and} \quad f(r) := \left(\frac{1}{\log 4} r^2 \log \frac{4}{r} \right)^\theta.$$

Using Lemma 3.9, it suffices to show

$$F \in L^p(B_1) \quad \text{and} \quad f^{-1} \in L^q(B_1).$$

First, consider $p \in (1, \infty)$. Using Fubini's theorem and radial coordinates, we obtain

$$\int_{B_1} |F(x)|^p dx \lesssim \int_0^1 dx_1 \int_0^1 r^{-2p(1-\theta)} (\log \frac{2}{r})^{-p(1-\theta)} r^{d-2} dr. \quad (3.20)$$

If the strict inequality $p < \frac{d-1}{2(1-\theta)}$ holds, we get $d-2-2p(1-\theta) > -1$ and (3.20) converges due to Lemma 2.8. Otherwise, if $p = \frac{d-1}{2(1-\theta)}$, we get $d-2-2p(1-\theta) = -1$. Because of $d \geq 4$, this case yields

$$-p(1-\theta) = -\frac{d-1}{2} < -1 \quad (3.21)$$

and (3.20) also converges due to Lemma 2.8. Now, consider $p = \infty$. This yields $\theta = 1$ and $F \equiv 1$. Therefore we have $F \in L^\infty(B_1)$.

Analogously, we get

$$\int_{B_1} |f_2^{-1}|^q dx \lesssim \int_0^1 r^{-2q\theta} (\log \frac{4}{r})^{-q\theta} r^{d-2} dr. \quad (3.22)$$

If $q < \frac{d-1}{2\theta}$, we get $d-2-2q\theta > -1$ and the integral converges by Lemma 2.8. Otherwise, if $q = \frac{d-1}{2\theta}$, we have $d-2-2q\theta = -1$. Because of $d \geq 4$, this case yields

$$-q\theta = -\frac{d-1}{2} < -1 \quad (3.23)$$

and (3.22) also converges due to Lemma 2.8. Now, consider $q = \infty$. This yields $\theta = 0$ and $f^{-1} \equiv 1$. Therefore we have $f^{-1} \in L^\infty(B_1)$. $\square$

Here we see why the borderline case applies in comparison to the spatial dimension $d = 3$. We benefit from the fact that the powers of the logarithm terms in (3.20) and in (3.22) are strictly less than -1 . That is why we are able to use Lemma 2.8. In comparison, for $d = 3$ the powers of the logarithm terms equal -1 which can be easily seen with (3.21) and (3.23). Thus, we cannot use Lemma 2.8 for this case.

Construction of the Subsolution

Now, we construct an unbounded weak subsolution of

$$\nabla \cdot (\omega_\theta \nabla u) = 0. \quad (3.24)$$

We define $\phi : (0, 1) \rightarrow \mathbb{R}^+$ and $v : B_1 \rightarrow \mathbb{R}^+$ such that

$$v(x) := e^{\alpha_d x_1} \phi(r), \quad (3.25)$$

where

$$\phi(r) := \begin{cases} i + \eta_i \beta_d (4^{-i(d-3)} r^{-(d-3)} - 1) & \text{when } r \in [\frac{1}{2} 4^{-i}, 4^{-i}), \\ i + 1 + (\eta_i - 1)(4^{i+1} r - 1)^2 & \text{when } r \in [4^{-i-1}, \frac{1}{2} 4^{-i}), \end{cases}$$

$$\beta_d := (2^{d-3} - 1)^{-1}, \quad \alpha_d := \sqrt{2^{d-1} \beta_d d(d-3)}, \quad (3.26)$$

$$\eta_i := \frac{2}{2 + 2^{d-4} \beta_d (d-3)(i+1) 4^{-2i}}. \quad (3.27)$$

For $d = 4$ we have $\beta_4 = 1$, $\alpha_4 = 4\sqrt{2}$ and $\eta_i = \frac{2}{2+(i+1)4^{-2i}}$ which corresponds to the definition of v in [FSC98], except that α_4 there equals 8. As before, η_i and α_d are essential for proving that v is a subsolution on a cylindrical shell. As we will see, our choice of α_d is optimal, i.e for every $\alpha'_d \geq \alpha_d$ the function $v(x) = e^{\alpha'_d x_1} \phi(r)$ is a subsolution of (3.24). Therefore, it is possible to choose $\alpha_4 = 8$ as it was done in [FSC98]. Furthermore, we introduce β_d to guarantee that ϕ , and therefore v , is continuous.

We continue as in section 3.1 by proving estimates for ω_θ and for the derivatives

$$\phi'(r) = \begin{cases} -(d-3)\eta_i \beta_d 4^{-i(d-3)} r^{-d+2} & \text{when } r \in (\frac{1}{2} 4^{-i}, 4^{-i}), \\ 2(\eta_i - 1) 4^{i+1} (4^{i+1} r - 1) & \text{when } r \in (4^{-i-1}, \frac{1}{2} 4^{-i}), \end{cases}$$

$$\phi''(r) = \begin{cases} (d-3)(d-2)\eta_i \beta_d 4^{-i(d-3)} r^{-d+1} & \text{when } r \in (\frac{1}{2} 4^{-i}, 4^{-i}), \\ 2(\eta_i - 1) 4^{2i+2} & \text{when } r \in (4^{-i-1}, \frac{1}{2} 4^{-i}). \end{cases}$$

Lemma 3.11. *Let ω_θ be as in (3.2) and ϕ be as in (3.8). Then,*

$$\omega_\theta(r) \lesssim \begin{cases} r^{2\theta} (\log \frac{4}{r})^\theta & \text{when } r \in [\frac{1}{2} 4^{-i}, 4^{-i}), \\ r^{-2(1-\theta)} (\log \frac{2}{r})^{-(1-\theta)} & \text{when } r \in [4^{-i-1}, \frac{1}{2} 4^{-i}), \end{cases}$$

$$|\phi(r)| \lesssim \begin{cases} \log \frac{4}{r} & \text{when } r \in (\frac{1}{2} 4^{-i}, 4^{-i}), \\ \log \frac{2}{r} & \text{when } r \in (4^{-i-1}, \frac{1}{2} 4^{-i}), \end{cases}$$

$$|\phi'(r)| \lesssim \begin{cases} r^{-1} & \text{when } r \in (\frac{1}{2} 4^{-i}, 4^{-i}), \\ r \log \frac{2}{r} & \text{when } r \in (4^{-i-1}, \frac{1}{2} 4^{-i}). \end{cases}$$

Proof. We proceed as in the proof of Lemma 3.11, but note that η_i and ϕ are defined differently. Fix $i \in \mathbb{N}_0$ and $r \in (\frac{1}{2} 4^{-i}, 4^{-i})$. Since $i+1 \leq \log \frac{4}{r}$ and $4^{-i} \leq 2r$, we get

$$\omega_\theta(r) = (i+1)^\theta 4^{-2i\theta} \lesssim r^{2\theta} (\log \frac{4}{r})^\theta.$$

Additionally, we obtain

$$|\phi(r)| = i + \eta_i \beta_d (4^{-i(d-3)} r^{-(d-3)} - 1) \leq i + 1 \leq \log \frac{4}{r}$$

and

$$|\phi'(r)| = (d-3)\eta_i\beta_d4^{-i(d-3)}r^{-d+2} \lesssim \frac{1}{r},$$

where we used $\eta_i \in (0, 1)$, $\beta_d(4^{-i(d-3)}r^{-(d-3)} - 1) \in [0, 1]$.

Now, fix $r \in (4^{-i-1}, \frac{1}{2}4^{-i})$. Since $i+1 \leq \log \frac{2}{r}$ and $4^{-i} \leq 4r$, we get

$$\omega_\theta(r) = (i+1)^{-(1-\theta)}4^{2i(1-\theta)} \lesssim r^{-2(1-\theta)}(\log \frac{2}{r})^{-(1-\theta)}.$$

Using $(\eta_i - 1) \in (-1, 0)$ and $(4^{i+1}r - 1)^2 \geq 0$, we obtain

$$|\phi(r)| = i+1 + (\eta_i - 1)(4^{i+1}r - 1)^2 \leq i+1 \leq \log \frac{2}{r}.$$

Now, we must argue for $\phi'(r)$ a bit differently than before. Using the definition of η_i in (3.27), we get

$$|\phi'(r)| = 2 \frac{2^{d-4}\beta_d(d-3)(i+1)4^{-2i}}{2 + 2^{d-4}\beta_d(d-3)(i+1)4^{-2i}} 4^{i+1}(4^{i+1}r - 1) \lesssim (i+1)4^{-2i}4^{i+1} \lesssim r \log \frac{2}{r},$$

where we needed $(4^{i+1}r - 1) \in [0, 1]$. □

As in section 3.1, we divide the proof that v is a subsolution of (3.24) into three steps.

Lemma 3.12. *Let v be as in (3.25). Then, $v \in H^1(\omega_\theta, B_1)$.*

Proof. This proof is almost identical to the one from Lemma 3.4. The only difference is that some integrals are slightly different due to the spherical transformation. To show $v \in H^1(\omega_\theta, B_1)$, it is sufficient to approximate it with Lipschitz functions with respect to the $H^1(\omega_\theta, B_1)$ -norm, since every Lipschitz function can be approximated by $C^\infty(B_1)$ functions. Define for $n \in \mathbb{N}$

$$\phi_n(r) := \phi(r)\mathbb{1}_{\{r \geq 4^{-n}\}} + n\mathbb{1}_{\{r < 4^{-n}\}} \quad \text{and} \quad v_n(x) := e^{\alpha x_1} \phi_n(r).$$

Each function v_n is Lipschitz-continuous and v_n and ∇v_n converge pointwise to v and ∇v , respectively. Moreover,

$$\begin{aligned} \|v - v_n\|_{H^1(\omega_\theta, B_1)} &= \int_{B_1} \omega_\theta(r) (|\nabla v(x) - \nabla v_n(x)|^2 + |v(x) - v_n(x)|^2) dx \\ &= \int_{B_1} e^{2\alpha x_1} \omega_\theta(r) \left((\alpha^2 + 1)|\phi(r) - n|^2 + |\phi'(r)|^2 \right) \mathbb{1}_{\{r < 4^{-n}\}}(r) dx \\ &\lesssim \int_0^1 \omega_\theta(r) (|\phi(r)|^2 + |\phi'(r)|^2) r^{d-2} dr, \end{aligned} \tag{3.28}$$

where we needed

$$\begin{aligned} |\nabla v(x) - \nabla v_n(x)| &= |\partial_1 v(x) - \partial_1 v_n(x)| + |\partial_r v(x) - \partial_r v_n(x)| \\ &= \alpha e^{2\alpha x_1} |\phi(r) - \phi_n(r)| + e^{2\alpha x_1} |\phi'(r) - \phi'_n(r)|, \end{aligned}$$

Fubini's theorem, radial coordinates and $\phi(r) \geq n$ if $r < 4^{-n}$.

If (3.28) converges, the claim follows by the dominated convergence theorem. We write

$$\begin{aligned} \int_0^1 \omega_\theta(r)(|\phi(r)|^2 + |\phi'(r)|^2) r^{d-2} dr &= \sum_{i=0}^{\infty} \int_{\frac{1}{2}4^{-i}}^{4^{-i}} \omega_\theta(r)(|\phi(r)|^2 + |\phi'(r)|^2) r^{d-2} dr \\ &\quad + \int_{4^{-i-1}}^{\frac{1}{2}4^{-i}} \omega_\theta(r)(|\phi(r)|^2 + |\phi'(r)|^2) r^{d-2} dr \end{aligned}$$

and use Lemma 3.11 to get

$$\begin{aligned} \sum_{i=0}^{\infty} \int_{\frac{1}{2}4^{-i}}^{4^{-i}} \omega_\theta(r)|\phi(r)|^2 r^{d-2} dr &\lesssim \int_0^1 (\log \frac{4}{r})^{2+\theta} r^{d-2+2\theta} dr, \\ \sum_{i=0}^{\infty} \int_{\frac{1}{2}4^{-i}}^{4^{-i}} \omega_\theta(r)|\partial_r \phi(r)|^2 r^{d-2} dr &\lesssim \int_0^1 (\log \frac{4}{r})^\theta r^{d-4+2\theta} dr, \\ \sum_{i=0}^{\infty} \int_{4^{-i-1}}^{\frac{1}{2}4^{-i}} \omega_\theta(r)|\phi(r)|^2 r^{d-2} dr &\lesssim \int_0^1 (\log \frac{2}{r})^{1+\theta} r^{d-4+2\theta} dr, \\ \sum_{i=0}^{\infty} \int_{4^{-i-1}}^{\frac{1}{2}4^{-i}} \omega_\theta(r)|\partial_r \phi(r)|^2 r^{d-2} dr &\lesssim \int_0^1 (\log \frac{2}{r})^{1+\theta} r^{d-2+2\theta} dr. \end{aligned}$$

All these integrals converge due to Lemma 2.8. $\square$

Lemma 3.13. *Let v be as in (3.25) and $\gamma_{2i} := 4^{-i}$, $\gamma_{2i+1} := \frac{1}{2}4^{-i}$ for $i \in \mathbb{N}_0$. Then, v is a weak subsolution of (3.24) in each cylindrical shell $B_1 \cap \{\gamma_N < r < 1\}$ with $N \in \mathbb{N}$.*

Proof. We proceed as in the proof of Lemma 3.5. Fix $N \in \mathbb{N}$ and denote $\Omega := B_1 \cap \{\gamma_N < r < 1\}$. Furthermore, let $\varphi \in C_c^\infty(\Omega)$ be nonnegative. Additionally, we define the cylindrical shells with

$$M_{2i} := B_1 \cap \{\tfrac{1}{2}4^{-i} < r < 4^{-i}\} \quad \text{and} \quad M_{2i+1} := B_1 \cap \{4^{-i-1} < r < \tfrac{1}{2}4^{-i}\},$$

the left-sided weighted derivative with $(\omega_\theta \phi')(r^-)$ and the right-sided weighted derivative with $(\omega_\theta \phi')(r^+)$. Since ω_θ is constant on each M_k , we obtain via integration by parts

$$\begin{aligned} \int_{B_1} \omega_\theta \nabla v \cdot \nabla \varphi &= \sum_{k=0}^N \int_{M_k} \omega_\theta \nabla v \cdot \nabla \varphi \\ &= \sum_{k=0}^N - \int_{M_k} \omega_\theta \Delta v \varphi + \int_{\partial M_k} \omega_\theta \nabla v \cdot \vec{n} \varphi \\ &= \sum_{k=0}^N - \int_{M_k} \omega_\theta \Delta v \varphi \\ &\quad + \sum_{k=1}^{N-1} \int_{B_1 \cap \{r=\gamma_k\}} e^{\alpha x_1} ((\omega_\theta \phi')(\gamma_k^-) - (\omega_\theta \phi')(\gamma_k^+)) \varphi(x) dx, \end{aligned}$$

3 Construction of the Counterexample

where $\vec{n}$ denotes the unit surface normal. Since φ is compactly supported in Ω , the boundary integrals on the sets $B_1 \cap \{r = 1\}$ and $B_1 \cap \{r = \gamma_N\}$ vanish. Thus, it is sufficient to prove

- (i) $-\Delta v \leq 0$ in M_k ,
- (ii) $(\omega_\theta \phi')(\gamma_k^-) - (\omega_\theta \phi')(\gamma_k^+) \leq 0$ for every k .

Proof of (i). Using spherical transformation, we obtain

$$-\Delta v(x) = -e^{\alpha_d x_1} \left(\alpha_d^2 \phi(r) + \frac{d-2}{r} \phi'(r) + \phi''(r) \right),$$

hence $-\Delta v \leq 0$ in M_k provides

$$\alpha_d^2 \phi(r) + \frac{d-2}{r} \phi'(r) + \phi''(r) \geq 0.$$

For $r \in (\frac{1}{2}4^{-i}, 4^{-i})$, we get

$$\begin{aligned} \alpha_d^2 \phi(r) + \frac{d-2}{r} \phi'(r) + \phi''(r) &= \alpha_d^2 (i + \eta_i \beta_d (4^{-i(d-3)} r^{-(d-3)} - 1)) \\ &\quad + \frac{d-2}{r} (-(d-3) \eta_i \beta_d 4^{-i(d-3)} r^{-(d-2)}) \\ &\quad + (d-3)(d-2) \eta_i \beta_d 4^{-i(d-3)} r^{-(d-1)} \\ &\geq \alpha_d^2 i \geq 0, \end{aligned}$$

where the second and third term cancel each other, and we used $\eta_i \beta_d (4^{-i(d-3)} r^{-(d-3)} - 1) \geq 0$.

Now, fix $r \in (4^{-i-1}, \frac{1}{2}4^{-i})$. Then, it follows that

$$\begin{aligned} \alpha_d^2 \phi(r) + \frac{d-2}{r} \phi'(r) + \phi''(r) & \\ = 2(\eta_i - 1)4^{2i+2} \left(\frac{\alpha_d^2(i+1)}{2(\eta_i - 1)4^{2i+2}} + \frac{\alpha_d^2(4^{i+1}r - 1)^2}{2 \cdot 4^{2i+2}} + \frac{d-2}{r}(r - 4^{-i-1}) + 1 \right). & \end{aligned} \tag{3.29}$$

Furthermore, we get

$$\begin{aligned}
 & \frac{\alpha_d^2(i+1)}{2(\eta_i-1)4^{2i+2}} + \frac{\alpha_d^2(4^{i+1}r-1)^2}{2 \cdot 4^{2i+2}} + \frac{d-2}{r}(r-4^{-i-1}) + 1 \\
 &= d-1 - \frac{4^{-i-1}(d-2)}{r} + \frac{\alpha_d^2}{2 \cdot 4^{2i+2}} \left(\frac{i+1}{\eta_i-1} + (4^{i+1}r-1)^2 \right) \\
 &\leq d-1 - \frac{1}{2}(d-2) - \frac{\alpha_d^2}{2 \cdot 4^{2i+2}} \left(\frac{i+1}{1-\eta_i} - 1 \right) \\
 &= \frac{d}{2} - \frac{\alpha_d^2}{2 \cdot 4^{2i+2}} \left(\frac{(i+1)(2+2^{d-4}\beta_d(d-3)(i+1)4^{-2i})}{2^{d-4}\beta_d(d-3)(i+1)4^{-2i}} - \frac{2^{d-4}\beta_d(d-3)4^{-2i}}{2^{d-4}\beta_d(d-3)4^{-2i}} \right) \\
 &= \frac{d}{2} - \frac{\alpha_d^2}{2 \cdot 4^{2i+2}} \frac{2+2^{d-4}\beta_d(d-3)i4^{-2i}}{2^{d-4}\beta_d(d-3)4^{-2i}} \\
 &= \frac{d}{2} - \frac{\alpha_d^2}{2^d\beta_d(d-3)} - \frac{\alpha_d^2 i}{2 \cdot 4^{2i+2}} \\
 &= -\frac{\alpha_d^2}{2} i 4^{-2i-2} \leq 0,
 \end{aligned}$$

where we used $-\frac{1}{r} \leq -2 \cdot 4^i$, $(4^{i+1}r-1)^2 \leq 1$ and the definitions of η_i in (3.25) and α_d in (3.26). This and $(\eta_i-1) \leq 0$ imply that the term (3.29) is greater than 0. In particular, note that α_d was defined so that the first and second summand in the penultimate line cancel each other. A greater value for α_d yields that (3.29) is even smaller.

Proof of (ii). Consider $r = 4^{-i}$. Then, the right-sided derivative satisfies

$$\phi'((4^{-i})^+) = \phi'((4^{-(i-1)-1})^+) = 2(\eta_{i-1}-1)4^i(4^i4^{-i}-1) = 0$$

and the left-sided derivative

$$\phi'((4^{-i})^-) = -(d-3)\eta_i\beta_d4^{-i(d-3)}4^{i(d-2)} < 0.$$

Thus, $(\omega_\theta\phi')((4^{-i})^-) - (\omega_\theta\phi')((4^{-i})^+) \leq 0$.

Now, fix $r = \frac{1}{2}4^{-i}$. Then, the right-sided weighted derivative is

$$\begin{aligned}
 (\omega_\theta\phi')\left(\left(\frac{1}{2}4^{-i}\right)^+\right) &= -(d-3)\eta_i\beta_d4^{-i(d-3)}2^{d-2}4^{i(d-2)}(i+1)^\theta4^{-2i\theta} \\
 &= -(d-3)\eta_i\beta_d2^{d-2}4^i(i+1)^\theta4^{-2i\theta}
 \end{aligned}$$

and the left-sided weighted derivative

$$\begin{aligned}
 (\omega_\theta\phi')\left(\left(\frac{1}{2}4^{-i}\right)^-\right) &= 2(\eta_i-1)4^{i+1}\left(\frac{1}{2}4-1\right)(i+1)^{-(1-\theta)}4^{2i(1-\theta)} \\
 &= -2\frac{2^{d-4}\beta_d(d-3)(i+1)4^{-2i}}{2+2^{d-4}\beta_d(d-3)(i+1)4^{-2i}}4^{i+1}(i+1)^{-(1-\theta)}4^{2i(1-\theta)} \\
 &= -(d-3)\eta_i\beta_d2^{d-2}4^i(i+1)^\theta4^{-2i\theta} \\
 &= (\omega_\theta\phi')\left(\left(\frac{1}{2}4^{-i}\right)^+\right).
 \end{aligned}$$

At this point we see that η_i is defined so that $(\omega_\theta\phi')\left(\left(\frac{1}{2}4^{-i}\right)^-\right) = (\omega_\theta\phi')\left(\left(\frac{1}{2}4^{-i}\right)^+\right)$. $\square$

Lemma 3.14. *Let v be as in (3.25). Then, v is a weak subsolution of (3.24) in B_1 .*

Proof. The only difference between the proof of Lemma 3.6 and this one is that some integrals are slightly different due to the spherical transformation. We define the cut off function $\psi_n : \mathbb{R}^+ \rightarrow [0, 1]$ such that $\psi_n \in C^1([0, 1])$, $\psi_n(r) = 0$ for $0 \leq r \leq \frac{1}{2}4^{-n}$, $\psi_n(r) = 1$ for $4^{-n} \leq r$ and $\psi'_n(r) \leq 4^{n+1}$. Consider a nonnegative $\varphi \in C_c^\infty(B_1)$. Then, we get

$$\int_{B_1} \nabla v \cdot \nabla \varphi = \int_{B_1} \omega_\theta \nabla v \cdot \nabla ((1 - \psi_n)\varphi) + \int_{B_1} \omega_\theta \nabla v \cdot \nabla (\psi_n \varphi).$$

Since $\psi_n \varphi$ has compact support with its support being a positive distance away from $\{x_1 = 0\}$, the second integral is negative according to Lemma 3.13. In the following, we argue that $\int_{B_1} \omega_\theta \nabla v \cdot \nabla ((1 - \psi_n)\varphi)$ is negligible, i.e. it vanishes as $n \rightarrow \infty$. This then completes the proof.

We get

$$\begin{aligned} \left| \int_{B_1} \omega_\theta \nabla v \cdot \nabla ((1 - \psi_n)\varphi) \right| &\leq \int_{B_1 \cap \{r < 4^{-n}\}} \omega_\theta |1 - \psi_n| |\nabla v \cdot \nabla \varphi| \\ &\quad + \int_{M_{2n}} \omega_\theta |\varphi| |\nabla v \cdot \nabla \psi_n|, \end{aligned} \quad (3.30)$$

where $M_{2n} = B_1 \cap \{\frac{1}{2}4^{-n} < r < 4^{-n}\}$. According to the dominated convergence theorem, the first integral in (3.30) converges to 0 as $n \rightarrow \infty$, since $\omega_\theta |1 - \psi_n| |\nabla v \cdot \nabla \varphi|$ converges pointwise to 0, and is dominated by $\omega_\theta |\nabla v \cdot \nabla \varphi|$, where, using Hölder inequality, we get

$$\int_{B_1} \omega_\theta |\nabla v \cdot \nabla \varphi| \leq \|\nabla \varphi\|_\infty \int_{B_1} \sqrt{\omega_\theta} \sqrt{\omega_\theta} |\nabla v| \leq \|\nabla \varphi\|_\infty \|\omega_\theta\|_1^{1/2} \|v\|_{H^1(\omega_\theta, B_1)} < \infty.$$

Then, we get for the second integral in (3.30)

$$\begin{aligned} \int_{M_{2n}} |\varphi| |\nabla v \cdot \nabla \psi_n| \omega_\theta &\leq \|\varphi\|_\infty \int_{M_{2n}} \omega_\theta |\nabla v \cdot 4^{n+1} \vec{e}_r| \\ &\lesssim \|\varphi\|_\infty \int_{\frac{1}{2}4^{-n}}^{4^{-n}} 4^{n+1} \omega_\theta(r) |\phi'| r^{d-2} dr \\ &\lesssim \|\varphi\|_\infty \int_{\frac{1}{2}4^{-n}}^{4^{-n}} (n+1)^\theta 4^{n(4-d-2\theta)} dr \\ &= \frac{1}{2} \|\varphi\|_\infty (n+1)^\theta 4^{n(3-d-2\theta)}, \end{aligned}$$

where we used Fubini's theorem, radial coordinates, the definition of ω_θ in (3.19), the estimate $|\phi'(r)| \lesssim \frac{1}{r}$ from Lemma 3.11 and $r \leq 2 \cdot 4^{-n}$. The sequence $(n+1)^\theta 4^{n(3-d-2\theta)}$ converges to 0 as $n \rightarrow \infty$. Therefore, the second integral in (3.30) also vanishes. $\square$

Existence of a solution

In the following, we solve the following Dirichlet problem

$$\begin{aligned} -\nabla \cdot \omega_\theta \nabla u &= 0 && \text{in } B_1, \\ u &= v && \text{on } \partial B_1, \end{aligned}$$

where v is the weak subsolution constructed above. This concludes the proof of Theorem 1.3 using Lemma 2.7. For that, we argue as in the construction for the subsolution. First, we show that on a positive distance away from the line $\{x_1 = 0\}$ a weak solution of (3.13) exists, i.e. a weak solution $\tilde{u}_n$ exists for

$$\begin{aligned} -\nabla \cdot \omega_\theta \nabla \tilde{u}_n &= 0 & \text{in } \Omega_n, \\ \tilde{u}_n &= v & \text{on } \partial\Omega_n, \end{aligned} \quad (3.31)$$

where $\Omega_n := B_1 \cap \{\frac{3}{8}4^{-n} < r < 1\}$ and $n \in \mathbb{N}$. Afterwards, we prove that the sequence $(\tilde{u}_n)_n$ converges to the desired weak solution of (3.13).

Lemma 3.15. *Let v be as in (3.25). Then, a weak solution $\tilde{u}_n \in H^1(\omega_\theta, \Omega_n)$ of (3.31) exists for each $n \in \mathbb{N}$. Additionally, the following estimate holds:*

$$\|\tilde{u}_n - v\|_{H^1(\omega_\theta, \Omega_n)} \leq \|v\|_{H^1(\omega_\theta, \Omega)}. \quad (3.32)$$

Proof. See proof of Lemma 3.7. $\square$

Lemma 3.16. *The Dirichlet problem (3.13) has a weak solution u .*

Proof. As before, the only difference between the proof of Lemma 3.8 and this one is that some integrals are slightly different due to the spherical transformation. Using Lemma 3.15, we obtain a sequence $(\tilde{u}_n)_n$ of weak solutions of the Dirichlet problem (3.31) for each $n \in \mathbb{N}$. We extend these functions as follows:

$$u_n := v + (\tilde{u}_n - v)\mathbb{1}_{\Omega_n}.$$

Since $(\tilde{u}_n - v)\mathbb{1}_{\Omega_n}$ is in $H_0^1(\omega_\theta, B_1)$ and $v \in H^1(\omega_\theta, B_1)$ due to Lemma 3.12, u_n belongs to $H^1(\omega_\theta, B_1)$. Using estimate (3.32), we get for every $n \in \mathbb{N}$

$$\|u_n\|_{H^1(\omega_\theta, B_1)} \leq 2\|v\|_{H^1(\omega_\theta, B_1)}. \quad (3.33)$$

Thus, a function $u \in H^1(\omega_\theta, B_1)$ exists such that

$$u_n \rightharpoonup u \quad \text{in } H^1(\omega_\theta, B_1).$$

Now, we have a candidate for a weak solution. Consider a $\varphi \in C_c^\infty(B_1)$. Then, the weak convergence implies

$$\int_{B_1} \omega_\theta \nabla u_n \cdot \nabla \varphi \xrightarrow{n \rightarrow \infty} \int_{B_1} \omega_\theta \nabla u \cdot \nabla \varphi.$$

The claim is proven if we show that $\int_{B_1} \omega_\theta \nabla u_n \cdot \nabla \varphi$ also converges to 0.

We define the cut off function $\psi_n : \mathbb{R}^+ \rightarrow [0, 1]$ such that $\psi_n \in C^1(\mathbb{R})$, $\psi_n(r) = 0$ for $0 \leq r \leq \frac{1}{2}4^{-n}$, $\psi_n(r) = 1$ for $4^{-n} \leq r$ and $\psi'_n(r) \leq 4^{n+1}$. Then, we get

$$\begin{aligned} \int_{B_1} \omega_\theta \nabla u_n \cdot \nabla \varphi &= \\ &= \int_{B_1 \cap \{r < \frac{3}{8}4^{-n}\}} \omega_\theta \nabla v \cdot \nabla \varphi + \int_{\Omega_n} \omega_\theta \nabla \tilde{u}_n \cdot \nabla ((1 - \psi_n)\varphi) + \int_{\Omega_n} \omega_\theta \nabla \tilde{u}_n \cdot \nabla (\psi_n \varphi). \end{aligned}$$

The first integral converges to 0 as shown in the proof of Lemma 3.14 and the last one vanishes, because $\tilde{u}_n$ is a weak solution in Ω_n and $\psi_n\varphi$ has compact support in Ω_n .

Let us observe the second integral which can be decomposed as follows:

$$\begin{aligned} \left| \int_{\Omega_n} \omega_\theta \nabla \tilde{u}_n \cdot \nabla ((1 - \psi_n)\varphi) \right| &\leq \int_{B_1 \cap \{r < 4^{-n}\}} \omega_\theta |1 - \psi_n| |\nabla \tilde{u}_n \cdot \nabla \varphi| \\ &\quad + \int_{M_{2n}} \omega_\theta |\varphi| |\nabla \tilde{u}_n \cdot \nabla \psi_n|, \end{aligned} \quad (3.34)$$

where $M_{2n} = B_1 \cap \{\frac{1}{2}4^{-n} < r < 4^{-n}\}$. According to the dominated convergence theorem, the first integral in (3.34) converges to 0 as $n \rightarrow \infty$, since $\omega_\theta |1 - \psi_n| |\nabla \tilde{u}_n \cdot \nabla \varphi|$ converges pointwise to 0, and is dominated by $\omega_\theta |\nabla \tilde{u}_n \cdot \nabla \varphi|$, where, using Hölder inequality and estimate (3.33), we get

$$\int_{B_1} \omega_\theta |\nabla \tilde{u}_n \cdot \nabla \varphi| \leq \|\nabla \varphi\|_\infty \int_{B_1} \sqrt{\omega_\theta} \sqrt{\omega_\theta} |\nabla u_n| \leq 2 \|\nabla \varphi\|_\infty \|\omega_\theta\|_{L^1}^{1/2} \|v\|_{H^1(\omega_\theta, B_1)}.$$

For the second integral in (3.34) we obtain

$$\begin{aligned} \int_{M_{2n}} \omega_\theta |\varphi| |\nabla \tilde{u}_n \cdot \nabla \psi_n| &\leq \|\varphi\|_\infty \|u_n\|_{H^1(\omega_\theta, B_1)} \left(\int_{M_{2n}} \omega_\theta |\nabla \psi_n|^2 \right)^{\frac{1}{2}} \\ &\lesssim 2 \|\varphi\|_\infty \|v\|_{H^1(\omega_\theta, B_1)} \left(\int_{\frac{1}{2}4^{-n}}^{4^{-n}} \omega_\theta(r) 4^{2n+2} r^{d-2} dr \right)^{\frac{1}{2}} \\ &\lesssim 2 \|\varphi\|_\infty \|v\|_{H^1(\omega_\theta, B_1)} \left(\int_{\frac{1}{2}4^{-n}}^{4^{-n}} (n+1)^\theta 4^{n(4-d-2\theta)} dr \right)^{\frac{1}{2}} \\ &= \|\varphi\|_\infty \|v\|_{H^1(\omega_\theta, B_1)} ((n+1)^\theta 4^{n(3-d-2\theta)})^{\frac{1}{2}}, \end{aligned}$$

where we used Hölder's inequality, estimate (3.33), Fubini's theorem, radial coordinates, the definition of ω_θ in (3.19) and $r \leq 2 \cdot 4^{-n}$, if $r \in (\frac{1}{2}4^{-n}, 4^{-n})$. Since the sequence $(n+1)^\theta 4^{n(3-d-2\theta)}$ converges to 0 as $n \rightarrow \infty$, the second integral in (3.34) also vanishes. $\square$

Bibliography

- [BS19] Peter Bella and Mathias Schäffner. “Local Boundedness and Harnack Inequality for Solutions of Linear Nonuniformly Elliptic Equations”. In: *Communications on Pure and Applied Mathematics* (2019).
- [De 57] Memoria di Ennio De Giorgi. “Sulla differenziabilità e l’analiticità delle estremali degli integrali multipli regolari”. In: *Ennio De Giorgi* (1957), p. 167.
- [Eva10] Lawrence C. Evans. *Partial differential equations*. 2. Edition. American Mathematical Society, 2010.
- [FSC98] Bruno Franchi, Raul Serapioni, and Francesco Serra Cassano. “Irregular solutions of linear degenerate elliptic equations”. In: *Potential Analysis* 9.3 (1998), pp. 201–216.
- [MS68] MRV Murthy and Guido Stampacchia. “Boundary value problems for some degenerate-elliptic operators”. In: *Annali di Matematica Pura ed Applicata* 80.1 (1968), pp. 1–122.
- [Nas58] John Nash. “Continuity of solutions of parabolic and elliptic equations”. In: *American Journal of Mathematics* 80.4 (1958), pp. 931–954.
- [Sch13] Ben Schweizer. *Partielle Differentialgleichungen*. 2. Auflage. Springer, 2013.
- [Tru71] Neil S Trudinger. “On the regularity of generalized solutions of linear, non-uniformly elliptic equations”. In: *Archive for Rational Mechanics and Analysis* 42.1 (1971), pp. 50–62.
- [Tru73] Neil S Trudinger. “Linear elliptic operators with measurable coefficients”. In: *Annali della Scuola Normale Superiore di Pisa-Classe di Scienze* 27.2 (1973), pp. 265–308.

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